

LOCALIZED BANDLIMITED NEARLY TIGHT FRAMES AND BESOV SPACES ON DOMAINS

Isaac Z. Pesenson¹

ABSTRACT. The goal of the present paper is to construct bandlimited highly localized and nearly tight frames on domains with smooth boundaries in Euclidean spaces. These frames are used to describe corresponding Besov spaces.

Dirichlet boundary conditions, second-order differential operators, eigenfunctions, frames, Besov spaces [2000] 43A85; 42C40; 41A17; Secondary 41A10

1. INTRODUCTION

The goal of the present paper is to construct bandlimited and highly localized almost tight frames on domains with smooth boundaries in Euclidean spaces. These frames are used to describe Besov spaces on domains.

Let $\Omega \subset \mathbb{R}^d$ be a domain with a smooth boundary Γ . In the space $L_2(\Omega)$ we consider a self-adjoint positive definite operator L generated by an expression

$$(1.1) \quad Lf = - \sum_{m,k=1}^d \partial_m(a_{m,k}(x)\partial_k f),$$

with zero boundary condition. Our main result is the following.

Theorem 1.1. (*Frame Theorem*) *For every $0 < \delta < 1$ there exists a set of functions $\Theta_{j,i} \in L_2(\Omega)$, $j \in [J, \infty)$, $1 \leq i \leq I_j$, such that:*

- (1) *$\{\Theta_{j,i}\}$ is a frame with constants $1 - \delta$ and 1, i.e.*

$$(1 - \delta)\|f\|_{L_2(\Omega)}^2 \leq \sum_{j=J}^{\infty} \sum_{1 \leq i \leq I_j} |\langle f, \Theta_{j,i} \rangle|^2 \leq \|f\|_{L_2(\Omega)}^2, \quad f \in L_2(\Omega);$$

- (2) *every $\Theta_{j,i}$ is bandlimited in the sense that it is a linear combination of eigenfunctions of L with eigenvalues in $[2^{2j-2}, 2^{2j+4}]$;*
 (3) *functions $\Theta_{j,i}$ have very strong localization in the following sense: for any $N > 0$ there exists a $C(N)$ such, that*

$$(1.2) \quad |\Theta_{j,i}(x)| \leq C(N) \sup_{y \in U_{j,i}} \frac{2^{dj}}{\max(1, 2^j|x-y|)^N},$$

where $\{U_{j,i}\}$ is a cover of Ω by sets whose diameter is comparable to $\delta^{1/d}2^{-j-2}$, $j \in [J, \infty)$

¹ Department of Mathematics, Temple University, Philadelphia, PA 19122; pesenson@temple.edu. The author was supported in part by the National Geospatial-Intelligence Agency University Research Initiative (NURI), grant HM1582-08-1-0019.

In Theorem 5.2 this result is used to describe Besov norm of a function $f \in L_2(\Omega)$ in terms of frame coefficients $\langle f, \Theta_{j,i} \rangle$.

We don't discuss any reconstruction method of a function $f \in L_2(\Omega)$ from its projections $\langle f, \Theta_{j,i} \rangle$. However, since our frame is "nearly" tight (at least when δ is close to zero) in practice one can use the same frame for reconstruction to have $f \approx \sum_{j,i} \langle f, \Theta_{j,i} \rangle \Theta_{j,i}$. Another way for reconstruction is to use iterative the so called frame algorithm, which in this case will exhibit geometric convergence with factor $\delta^n(2 - \delta)^{-n}$, where n is the number of iteration steps. Another possibility for reconstruction is through interpolation by variational splines exactly as it was done in [32]-[35].

In section 3 we construct frames in spaces of bandlimited functions $E_\omega(L) = \text{span}\{u_k\}$, $Lu_k = \lambda_k u_k$, $\lambda_k \leq \omega$, in a way that their frame constants are independent on ω (Theorem 3.3). It is important to note that according to our conditions a number of "samples" $\Phi_i(f)$ is approximately $|\Omega|\omega^{d/2}$, which according to the Weyl's asymptotic formula [19], [43], is essentially the dimension of the space $E_\omega(\Omega)$. In this sense Theorem 3.3 is optimal.

In section 4 we represent functions in terms of appropriate bandlimited components and apply Theorem 3.3. Localization of frame elements follows from well known properties of spectral projectors for self-adjoint second-order differential operators on manifolds [19], [43]. In section 5 we introduce Besov spaces as interpolation spaces between $L_2(\Omega)$ and domains of powers of L and characterize them in terms of frame coefficients. Results similar to Theorem 5.1 can be found in [39], [40]. A result, which is more general than Theorem 5.1 will appear in [28].

The most important fact for our construction of frames is that in a space of ω -bandlimited functions $E_\omega(L)$ the continuous and discrete norms are equivalent. This result in the case of compact and non-compact manifolds of bounded geometry was first discovered and explored in many ways in our papers [31]-[39]. In the classical cases of straight line \mathbb{R} and circle \mathbb{S} the corresponding results are known as Plancherel-Polya and Marcinkiewicz-Zygmund inequalities. Our generalization of Plancherel-Polya and Marcinkiewicz-Zygmund inequalities implies that ω -bandlimited functions on manifolds of bounded geometry are completely determined by the values of their averages over "small" sets "uniformly" distributed over M with a spacing comparable to $1/\sqrt{\omega}$ and can be completely reconstructed in a stable way from such sets of values. The last statement is an extension of the famous Shannon sampling theorem to the case of Riemannian manifolds of bounded geometry.

The present paper is the first systematic development of bandlimited localized frames and their relations to Besov spaces on general domains. Several approaches to frames on the unit ball in \mathbb{R}^d were considered in [42], [20], [21] but their methods and results are very different from ours.

Most of our proofs and results hold for general compact Riemannian manifolds without boundary and even for non-compact manifolds of bounded geometry. We do not discuss such manifolds in this paper since for, say, compact closed manifolds nearly tight bandlimited and localized frames were already developed in [13]. In the case of homogeneous compact manifolds bandlimited and localized tight frames were constructed in [17]. In the following papers a number of frames was constructed in different function spaces on closed compact manifolds and on non-compact manifolds [3], [4]-[6], [9]-[18], [27]-[41]. On compact manifolds necessary

conditions for sampling and interpolation in terms of Beurling-Landau densities were obtained in [26], [30].

Applications of frames on manifolds to scattering theory, statistic and cosmology can be found in [2], [15], [16], [18], [25], [24], [20], [21].

Acknowledgement: The relevance of frames on manifolds to cosmology was discussed with Domenico Marinucci during my visit to the University of Rome Tor Vergata in May 2012. I would like to thank him for inviting me and for numerous conversations, which stimulated my interest to frames on domains in Euclidean spaces.

2. BOUNDED DOMAINS WITH SMOOTH BOUNDARIES AND OPERATORS

We consider bounded domains $\Omega \subset \mathbb{R}^d$ with a smooth boundaries Γ which are smooth $(d-1)$ -dimensional oriented manifolds. Let $\overline{\Omega} = \Omega \cup \Gamma$ and $L_2(\Omega)$ be the space of functions square-integrable with respect to Lebesgue measure $dx = dx_1 \dots dx_d$ with the norm denoted as $\|\cdot\|$. If k is a natural number the notations $H^k(\Omega)$ will be used for the Sobolev space of distributions on Ω with the norm

$$\|f\|_{H^k(\Omega)} = \left(\|f\|^2 + \sum_{1 \leq |\alpha| \leq k} \|\partial^{|\alpha|} f\|^2 \right)^{1/2}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\partial^{|\alpha|}$ is a mixed partial derivative

$$\left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_d} \right)^{\alpha_d}.$$

Under our assumptions the space $C_0^\infty(\overline{\Omega})$ of infinitely smooth functions with support in $\overline{\Omega}$ is dense in $H^k(\Omega)$. Closure in $H^k(\Omega)$ of the space $C_0^\infty(\Omega)$ of smooth functions with support in Ω will be denoted as $H_0^k(\Omega)$.

Since Γ can be treated as a smooth Riemannian manifold one can introduce Sobolev scale of spaces $H^s(\Gamma)$, $s \in \mathbb{R}$, as, for example, the domains of the Laplace-Beltrami operator \mathcal{L} of a Riemannian metric on Γ .

According to the trace theorem there exists a well defined continuous surjective trace operator

$$\gamma : H^s(\Omega) \rightarrow H^{s-1/2}(\Gamma), \quad s > 1/2,$$

such that for all functions f in $H^s(\Omega)$ which are smooth up to the boundary the value γf is simply a restriction of f to Γ .

One considers the operator (1.1) with coefficients in $C^\infty(\Omega)$ where the matrix $(a_{j,k}(x))$ is real, symmetric and positive definite on $\overline{\Omega}$. The operator L is defined as the Friedrichs extension of L , initially defined on $C_0^\infty(\Omega)$, to the set of all functions f in $H^2(\Omega)$ with constrain $\gamma f = 0$. The Green formula implies that this operator is self-adjoint. The domain of its positive square root $L^{1/2}$ is the set of all functions f in $H^1(\Omega)$ for which $\gamma f = 0$.

Thus, one obtains a self-adjoint positive definite operator in the Hilbert space $L_2(\Omega)$ with a discrete spectrum $0 < \lambda_1 \leq \lambda_2, \dots$ which goes to infinity.

3. AVERAGE SAMPLING AND BANDLIMITED FRAMES ON DOMAINS

Let $Q(\rho), Q(2\rho) \subset \mathbb{R}^d$ be standard cubes of diameters ρ and 2ρ respectively with centers at zero. Let $U \subset Q(\rho)$ be a closed set and $d\mu$ be a positive measure on U .

We will assume that the total measure of U is finite and not zero, i.e.

$$0 < |U| = \int_U d\mu < \infty.$$

We consider the following distribution on $C^\infty(Q(\rho))$,

$$(3.1) \quad \Psi(\varphi) = \frac{1}{|U|} \int_U \varphi d\mu, \quad |U| = \int_U d\mu, \quad \varphi \in C_0^\infty(Q(\rho)).$$

Some examples of such distributions which are of particular interest to us are the following.

1) Weighted Dirac measures. In this case $U = \{x\}$, $x \in Q(\rho)$, measure $d\mu$ is any non-zero number μ and $\Psi(f) = \mu\delta_x(f) = \mu f(x)$.

2) Finite or infinite sequences of Dirac measures $\delta_j, x_j \in Q(\rho)$, with corresponding weights μ_j . In this case $U = \bigcup_j \{x_j\}$ and

$$\Psi(f) = \sum_j \mu_j \delta_{x_j}(f),$$

where we assume the following

$$0 < |U| = \sum_j |\mu_j| < \infty, \quad U = \bigcup_j \{x_j\}.$$

3) U is a smooth submanifold in $Q(\rho)$ of any codimension and $d\mu$ is its "surface" measure.

4) U is a measurable subset of $Q(\rho)$, $d\mu$ is the Lebesgue measure dx , and $|U| \neq 0$.

The following statement is an analog of the Poincaré inequality.

Lemma 3.1. *For any $k > d - 1$ there exists a constant $C = C(d, k) > 0$ such that the following inequality holds true*

$$(3.2) \quad \|f - \Psi(f)\|_{L_2(U)}^2 \leq C(d, k) \sum_{1 \leq |\alpha| \leq k} \rho^{2|\alpha|} \|\partial^{|\alpha|} f\|_{L_2(Q(2\rho))}^2,$$

for all $f \in H^k(\Omega)$, $k > d/2$, where $\alpha = (\alpha_1, \dots, \alpha_d)$, and $\partial^{|\alpha|} f = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f$ is a partial derivative of order $|\alpha| = \alpha_1 + \dots + \alpha_d$.

Proof. For any $f \in C^\infty(\Omega)$ and every $x, y \in U \subset Q(\rho)$, we have the following

$$(3.3) \quad \begin{aligned} f(x) &= f(y) + \sum_{1 \leq |\alpha| \leq k-1} \frac{1}{\alpha!} \partial^{|\alpha|} f(y) (x-y)^\alpha + \\ &\quad \sum_{|\alpha|=k} \frac{1}{\alpha!} \int_0^\eta t^{k-1} \partial^{|\alpha|} f(y+t\vartheta) \vartheta^\alpha dt, \end{aligned}$$

where $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $(x-y)^\alpha = (x_1-y_1)^{\alpha_1} \dots (x_d-y_d)^{\alpha_d}$, $\eta = \|x-y\|$, $\vartheta = (x-y)/\eta$.

We integrate over $U \subset Q(\rho)$ with respect to $d\mu(y)$. It gives

$$\begin{aligned} f(x) - \Psi(f) &= |U|^{-1} \int_U \left(\sum_{1 \leq |\alpha| \leq k-1} \frac{1}{\alpha!} \partial^{|\alpha|} f(y) (x-y)^\alpha \right) d\mu(y) + \\ &\quad |U|^{-1} \int_U \left(\sum_{|\alpha|=k} \frac{1}{(k-1)!} \int_0^\eta t^{k-1} \partial^{|\alpha|} f(y+t\vartheta) \vartheta^\alpha dt \right) d\mu(y). \end{aligned}$$

From here we obtain

$$\|f - \Psi(f)\|_{L_2(U)} \leq C(k)|U|^{-1} \sum_{1 \leq |\alpha| \leq k-1} \left(\int_U \left(\int_U |\partial^{|\alpha|} f(y)(x-y)^\alpha| d\mu(y) \right)^2 dx \right)^{1/2} +$$

(3.4)

$$C(k)|U|^{-1} \sum_{|\alpha|=k} \left(\int_U \left(\int_U \left| \int_0^\eta t^{k-1} \partial^{|\alpha|} f(y+t\vartheta) \vartheta^\alpha dt \right| d\mu(y) \right)^2 dx \right)^{1/2} = I + II.$$

Note that the following inequality holds (see [27], Ch.V) for any $\varphi \in C^\infty(Q(2\rho))$

$$(3.5) \quad \sup_{x \in B_i(\rho/2)} |\varphi(x)| \leq C(d, k) \sum_{0 \leq j \leq k} \rho^{j-d/2} \|\varphi\|_{H^j(Q(2\rho))}, \quad k > d/2,$$

which implies the estimate for any $\varphi \in C^\infty(Q(2\rho))$

$$(3.6) \quad |\Psi(\varphi)| \leq \sup_{x \in Q(\rho)} |\varphi(x)| \leq C(d, k) \sum_{0 \leq j \leq k} \rho^{j-d/2} \|\varphi\|_{H^j(Q(2\rho))}, \quad k > d/2.$$

Since $U \subset Q(\rho)$, $x, y \in U$, one has $x - y \in Q(2\rho)$, and then by (3.6) for every $|\alpha| \leq k-1$ we have

$$\left(\int_U |\partial^{|\alpha|} f(y)(x-y)^\alpha| d\mu(y) \right)^2 \leq C(d, k) \rho^{2|\alpha|} \sum_{j \leq k} \rho^{2j-d} \|\partial^{|\alpha|} f\|_{H^j(B(2\rho))}^2.$$

Thus, we obtain that

$$(3.7) \quad I \leq C(d, k) \sum_{1 \leq |\gamma| \leq m-1} \rho^{|\gamma|} \|\partial^{|\gamma|} f\|_{L_2(Q(2\rho))}, \quad m > d-1.$$

By the Schwartz inequality using the assumption $k > d/2$ one can obtain the following inequality

$$\left| \int_0^\eta t^{k-1} \partial^{|\alpha|} f(y+t\vartheta) \vartheta^\alpha dt \right| \leq C \eta^{k-d/2} \left(\int_0^\eta t^{d-1} |\partial^{|\alpha|} f(y+t\vartheta)|^2 dt \right)^{1/2}.$$

Thus, the Minkovski inequality gives that

$$II \leq C(k)|U|^{-1} \sum_{|\alpha|=k} \int_U \left(\int_U \left| \int_0^\eta t^{k-1} \partial^{|\alpha|} f(y+t\vartheta) \vartheta^\alpha dt \right|^2 dx \right)^{1/2} d\mu(y) \leq$$

$$C(k)|U|^{-1} \sum_{|\alpha|=k} \left(\int_U \left(\int_U \eta^{2k-d} \int_0^\eta t^{d-1} |\partial^{|\alpha|} f(y+t\vartheta)|^2 dt \right) dx \right)^{1/2} d\mu(y).$$

We integrate over $Q(\rho)$ using the spherical coordinate system (η, ϑ) . Since $\eta \leq \rho$ for $|\alpha| = k$ we obtain

$$(3.8) \quad \int_0^{\rho/2} \eta^{d-1} \int_0^{2\pi} \left| \int_0^\eta t^{k-1} \partial^k f(y+t\vartheta) \vartheta^\alpha dt \right|^2 d\vartheta d\eta \leq$$

$$C \int_0^{\rho/2} t^{d-1} \left(\int_0^{2\pi} \int_0^\rho \eta^{2k-d} |\partial^k f(y+t\vartheta)|^2 \eta^{d-1} d\eta d\vartheta \right) dt \leq C(\Omega, k) \rho^{2k} \|\partial^k f\|_{L_2(Q(2\rho))}^2.$$

The result follows from (3.7) and (3). \square

We consider a regular cover of \mathbb{R}^d by closed disjoint cubes $\{Q_\nu(\rho)\}$ of diameter ρ . Thus, two cubes from this family can intersect only over their boundaries. Set $U_\nu(\rho) = Q_\nu(\rho) \cap \Omega$ and let $\{U_i(\rho)\}$ be a subcollection of all $U_\nu(\rho)$ which have positive measure. Obviously, the collection $\{U_i(\rho)\}$ is a cover of Ω and $\text{diam } U_i(\rho) \leq \rho$. Thus,

$$(3.9) \quad U_i = U_i(\rho) = Q_i(\rho) \cap \Omega, \quad \bigcup_i U_i = \Omega, \quad \text{diam } U_i(\rho) \leq \rho.$$

Next, we introduce a family $\Psi = \{\Psi_i\}$ of functionals on $L_2(\Omega)$ where every functional has the form

$$(3.10) \quad \Psi_i(f) = \frac{1}{|U_i|} \int_{U_i} f(x) dx, \quad f \in L_2(\Omega), \quad |U_i| = \int_{U_i} dx.$$

Note, that in general the functionals Ψ_i are continuous only on the space of smooth functions (see (3.6)). However, if $d\mu_i(x) = dx$ and every U_i is open, then the Schwartz inequality gives continuity of Ψ_i on $L_2(\Omega)$. Our global Poincare inequality is the following.

Lemma 3.2. *For any $k > d-1$ there exist constants $c = c(\Omega, L, k)$, $C = C(\Omega, L, k)$ such that for any given $0 < \delta < 1$ if $\rho < c\delta$ then the following inequality holds*

$$(3.11) \quad (1 - 2\delta/3) \|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i| |\Psi_i(f)|^2 + C\rho^{2k}\delta^{-1} \|L^{k/2}f\|_{L_2(\Omega)}^2$$

for all $f \in \mathcal{D}(L^{k/2})$.

Proof. We will need the inequality (3.12) below. One has for all $\alpha > 0$

$$|A|^2 = |A - B|^2 + 2|A - B||B| + |B|^2, \quad 2|A - B||B| \leq \alpha^{-1}|A - B|^2 + \alpha|B|^2,$$

which imply the inequality

$$(1 + \alpha)^{-1}|A|^2 \leq \alpha^{-1}|A - B|^2 + |B|^2, \quad \alpha > 0.$$

If, in addition, $0 < \alpha < 1$, then one has

$$(3.12) \quad (1 - \alpha)|A|^2 \leq \frac{1}{\alpha}|A - B|^2 + |B|^2, \quad 0 < \alpha < 1.$$

Applying inequality (3.12) we obtain

$$(3.13) \quad (1 - \alpha) \|f\|_{L_2(\Omega)}^2 \leq \sum_i (1 - \alpha) \|f\|_{L_2(U_i)}^2 \leq \alpha^{-1} \sum_i \|f - \Psi_i(f)\|_{L_2(U_i)}^2 + \sum_i |U_i| |\Psi_i(f)|^2, \quad |U_i| = \int_{U_i} dx.$$

Since Ω has a smooth boundary, there exist a linear continuous extension operator (see [22], Sec. 8.1)

$$H^k(\Omega) \rightarrow H^k(\mathbb{R}^d), \quad f \rightarrow \tilde{f} \in H^k(\mathbb{R}^d).$$

Note, (see [19], Sec. 17.5), that the following continuous embedding holds $\mathcal{D}(L^{k/2}) \subset H^k(\Omega)$, $k \in \mathbb{N}$, holds, where $\mathcal{D}(L^{k/2})$ is considered with the graph norm.

Thus, if $f \in \mathcal{D}(L^{k/2})$, then according to Lemma 3.1 one has for every i :

$$\|f - \Psi_i(f)\|_{L_2(U_i)}^2 = \|\tilde{f} - \Psi_i(\tilde{f})\|_{L_2(U_i)}^2 \leq C(d, k) \sum_{1 \leq |\alpha| \leq k} \rho^{2|\alpha|} \|\partial^{|\alpha|} \tilde{f}\|_{L_2(Q_i(2\rho))}^2.$$

Applying (3) with $\alpha = \delta/3$ and summing over i we obtain the following

$$(1 - \delta/3)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i|\|\Psi_i(f)\|^2 + \frac{3C(\Omega, k)}{\delta} \sum_{1 \leq j \leq k} \rho^{2j} \|\tilde{f}\|_{H^j(\cup_i Q_i(2\rho))}^2.$$

Since there exists a $C(\Omega, k)$ such that for all $1 \leq j \leq k$

$$\|\tilde{f}\|_{H^j(\cup_i Q_i(2\rho))}^2 \leq \|\tilde{f}\|_{H^j(\mathbb{R}^d)}^2 \leq C(\Omega, k)\|f\|_{H^j(\Omega)}^2$$

we obtain

$$(1 - \delta/3)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i|\|\Psi_i(f)\|^2 + \frac{C'(\Omega, k)}{\delta} \sum_{1 \leq j \leq k} \rho^{2j} \|f\|_{H^j(\Omega)}^2.$$

The regularity theorem for the elliptic second-order differential operator L (see [19], Sec. 17.5)

$$(3.14) \quad \|f\|_{H^j(\Omega)}^2 \leq b \left(\|f\|_{L_2(\Omega)}^2 + \|L^{j/2} f\|_{L_2(\Omega)}^2 \right), \quad f \in \mathcal{D}(L^{k/2}), \quad b = b(\Omega, L, j),$$

and the following interpolation inequality (see [19], Sec. 17.5)

$$(3.15) \quad \rho^{2j} \|L^{j/2} f\|_{L_2(\Omega)}^2 \leq 4a^{k-j} \rho^{2k} \|L^{k/2} f\|_{L_2(\Omega)}^2 + ca^{-j} \|f\|_{L_2(\Omega)}^2, \quad c = c(\Omega, L, k),$$

which holds for any $a, \rho > 0$, $0 \leq j \leq k$, imply that there exists a constant $C'' = C''(\Omega, L, k)$ such that the next inequality takes place

$$(1 - \delta/3)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i|\|\Psi_i(f)\|^2 +$$

$$C'' \left(\rho^2 \delta^{-1} \|f\|_{L_2(\Omega)}^2 + \rho^{2k} \delta^{-1} \|L^{k/2} f\|_{L_2(\Omega)}^2 + a^{-1} \|f\|_{L_2(\Omega)}^2 \right)$$

where $k > d/2$. By choosing $a = (6C''/\delta) > 1$ we obtain, that there exists a constant $C''' = C'''(\Omega, L, k)$ such that for any $0 < \delta < 1$ and $\rho > 0$

$$(1 - \delta/2)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i|\|\Psi_i(f)\|^2 + C''' \left(\rho^2 \delta^{-1} \|f\|_{L_2(\Omega)}^2 + \rho^{2k} \delta^{-1} \|L^{k/2} f\|_{L_2(\Omega)}^2 \right).$$

The last inequality shows, that if for a given $0 < \delta < 1$ the value of ρ is choosen such that

$$\rho < c\delta, \quad c = \frac{1}{\sqrt{6C'''}} \quad C''' = C'''(\Omega, L, k),$$

then we obtain for a $k > d/2$

$$(1 - 2\delta/3)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i|\|\Psi_i(f)\|^2 + C''' \delta^{-1} \rho^{2k} \|L^{k/2} f\|_{L_2(\Omega)}^2.$$

Lemma is proved. \square

In the space $L_2(M)$ we consider the functionals

$$\Phi_i(f) = \sqrt{|U_i|} \Psi_i(f) = \frac{1}{\sqrt{|U_i|}} \int_{U_i} f(x) dx, \quad |U_i| = \int_{U_i} dx.$$

Since the functionals $\Phi_i(f)$ are continuous on a subspace $E_\omega(L)$ they can be identified with certain functions in $E_\omega(L)$. The theorem below shows that the corresponding set of functions is a frame in appropriate subspace of bandlimited functions.

Theorem 3.3. *There exists a $c = c(\Omega, L)$ such that, if for a given $0 < \delta < 1$ and an $\omega > 0$ one has $\rho < c\delta^{1/d}\omega^{-1/2}$, and conditions (3.9) and (3.10) are satisfied, then*

$$(3.16) \quad (1 - \delta)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |\Phi_i(f)|^2 \leq \|f\|_{L_2(\Omega)}^2, \quad 0 < \delta < 1, \quad f \in E_\omega(L).$$

Proof. By using the Schwartz inequality we obtain the right-hand side of (3.16)

$$\sum_i |U_i| |\Psi_i(f)|^2 = \sum_i \frac{|U_i|}{|U_i|^2} \left| \int_{U_i} f dx \right|^2 \leq \sum_i \int_{U_i} |f|^2 dx = \|f\|_{L_2(\Omega)}^2, \quad f \in L_2(\Omega).$$

According to the previous lemma, there exist $c = c(\Omega, L)$, $C = C(\Omega, L)$ such that for any $0 < \delta < 1$ and any $\rho < c\delta$

$$(3.17) \quad (1 - \delta/2)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i| |\Psi_i(f)|^2 + C\rho^{2d}\delta^{-1}\|L^{d/2}f\|_{L_2(\Omega)}^2.$$

Notice, that if $f \in E_\omega(L)$, then the Bernstein inequality holds

$$(3.18) \quad \|L^{d/2}f\|_{L_2(\Omega)}^2 \leq \omega^d \|f\|_{L_2(\Omega)}^2.$$

Inequalities (3.17) and (3.18) show that for a certain $c = c(\Omega, L)$, if $\rho < c\delta^{1/d}\omega^{-1/2}$, then

$$(3.19) \quad (1 - \delta)\|f\|_{L_2(\Omega)}^2 \leq \sum_i |U_i| |\Psi_i(f)|^2, \quad 0 < \delta < 1, \quad f \in E_\omega(L).$$

Lemma is proved. \square

4. BANDLIMITED LOCALIZED FRAMES ON DOMAINS

4.1. Bandlimited frames. Choose a function $F \in C_c^\infty(\mathbb{R})$, supported in the interval $[2^{-2}, 2^4]$ such that

$$(4.1) \quad \sum_{j=-\infty}^{\infty} |F(2^{-2j}s)|^2 = 1$$

for all $s > 0$. For example, we could choose a smooth monotonically decreasing function τ on \mathbb{R}^+ with $0 \leq \tau \leq 1$, which is $\tau \equiv 1$ in $[0, 2^{-2}]$ and which is $\tau = 0$ in $[2^2, \infty)$. Then F can be defined as $F(s) = [\tau(s/2^2) - \tau(s)]^{1/2}$, $s > 0$. Let

$$J = \left\lceil -2 - \frac{1}{2} \log_2 \lambda_0 \right\rceil - 1,$$

where $\lambda_0 > 0$ is the first eigenvalue of the operator L and $\lceil \cdot \rceil$ greatest integer function. Using the spectral theorem for L one can obtain

$$(4.2) \quad \sum_{j=J}^{\infty} |F|^2(2^{-2j}L) = I.$$

where the sum (of operators) converges strongly on $L_2(\Omega)$. By applying both sides of this formula to an $f \in L_2(\Omega)$ and taking inner product with f gives

$$(4.3) \quad \sum_{j=J}^{\infty} \|F(2^{-2j}L)f\|_{L_2(\Omega)}^2 = \|f\|_{L_2(\Omega)}^2.$$

Moreover, since function $F(2^{-2j}\cdot)$ has support in $[2^{2j-2}, 2^{2j+4}]$ the function $F(2^{-2j}L)f$ is bandlimited to $[2^{2j-2}, 2^{2j+4}]$. We consider the sequence

$$\omega_j = 2^{2j+4}, \quad j = 0, 1, \dots,$$

and fix a $0 < \delta < 1$. For the constant $c = c(\Omega, L) > 0$ from Theorem 3.3 and for a fixed $0 < \delta < 1$ construct the sequence

$$\rho_j = c\delta^{1/d}\omega^{-1/2} = c\delta^{1/d}2^{-j-2}, \quad j = 0, 1, \dots$$

For any fixed $j = 0, 1, \dots$, any cover $\{U_{j,i}\}_{i=1}^{I_j}$ that satisfies (3.9), if $\{\Psi_{j,i}\}_{i=1}^{I_j}$ is the corresponding set of functionals constructed according to (3.10), then the frame inequalities (3.16) hold in every space $E_{\omega_j}(L)$. Set

$$\Phi_{j,i}(f) = \sqrt{|U_{j,i}|} \Psi_i(f),$$

then (3.16) imply for every $j \in [J, \infty]$

(4.4)

$$(1 - \delta) \|F(2^{-2j}L)f\|_{L_2(\Omega)}^2 \leq \sum_{i=1}^{I_j} |\langle F(2^{-2j}L)f, \Phi_{j,i} \rangle|^2 \leq \|F(2^{-2j}L)f\|_{L_2(\Omega)}^2,$$

where $F(2^{-2j}L)f \in E_{\omega_j}(L) = E_{2^{2j+4}}(L)$. Together with (4.3) it gives for any $f \in L_2(\Omega)$ the following inequalities

$$(4.5) \quad (1 - \delta) \|f\|_{L_2(\Omega)}^2 \leq \sum_{j=J}^{\infty} \sum_{i=1}^{I_j} |\langle F(2^{-2j}L)f, \Phi_{j,i} \rangle|^2 \leq \|f\|_{L_2(\Omega)}^2, \quad f \in L_2(\Omega).$$

Note, that since every functional $\Phi_{j,i}$ is continuous on $E_{\omega_j}(L)$ it can be identified with a function in $E_{\omega_j}(L)$, which still will be denoted as $\Phi_{j,i}$.

Since operator $F(2^{-2j}L)$ is self-adjoint, i.e. $\langle F(2^{-2j}L)f, \Phi_{j,i} \rangle = \langle f, F(2^{-2j}L)\Phi_{j,i} \rangle$, we obtain, that for

$$(4.6) \quad \Theta_{j,i} = F(2^{-2j}L)\Phi_{j,i} \in E_{\omega_j}(L),$$

the following double inequality holds for every $f \in L_2(\Omega)$

$$(4.7) \quad (1 - \delta) \|f\|_{L_2(\Omega)}^2 \leq \sum_{j=J}^{\infty} \sum_{i=1}^{I_j} |\langle f, \Theta_{j,i} \rangle|^2 \leq \|f\|_{L_2(\Omega)}^2, \quad f \in L_2(\Omega).$$

4.2. Localization of frame functions. According to the spectral Theorem if a self-adjoint positive-definite operator L has a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots$, and a corresponding set of eigenfunctions $\{u_j\}$, with $Lu_j = \lambda_j u_j$, which forms an orthonormal basis in $L_2(\Omega)$, then for any bounded real-valued function F of one variable one can construct a self-adjoint bounded operator $\mathcal{F}(L)$ in $L_2(\Omega)$ as

$$(4.8) \quad \mathcal{F}(L)f(x) = \int_{\Omega} K^{\mathcal{F}}(x, y)f(y)dy, \quad f \in L_2(\Omega),$$

where $K^{\mathcal{F}}(x, y)$ is a smooth function defined as

$$(4.9) \quad K^{\mathcal{F}}(x, y) = \sum_m \mathcal{F}(\lambda_m)u_m(x)\overline{u_m(y)}.$$

In what follows the following notations will be used

$$(4.10) \quad [\mathcal{F}(t^2L)f](x) = \int_{\Omega} K_t^{\mathcal{F}}(x, y)f(y)dy, \quad f \in L_2(\Omega),$$

where

$$(4.11) \quad K_t^{\mathcal{F}}(x, y) = \sum_m \mathcal{F}(t^2 \lambda_m) u_m(x) \overline{u_m(y)}.$$

Localization properties of the kernel $K_t^{\mathcal{F}}(x, y)$ are given in the following statement.

Lemma 4.1. *If L is an elliptic self-adjoint second order differential operators on compact manifolds (without boundary or with a smooth boundary) and $K_t^{\mathcal{F}}(x, y)$ is given by (4.9), then the following holds*

1) *If F is any Schwartz function on \mathbb{R} , then*

$$(4.12) \quad K_t^F(x, x) \sim c t^{-d}, \quad t \rightarrow 0.$$

2) *If $x \neq y$ and, in addition, F is even, then $K_t^F(x, y)$ vanishes to infinite order as t goes to zero.*

For the proof see [43], Sec.12.2, 12.3, Proposition 3.6, and also [19], sec. 17.5.

Since $K_t^{\mathcal{F}}(x, y)$ is smooth and Ω is bounded we can express localization of $K_t^{\mathcal{F}}(x, y)$ by using the following inequality: for any $N > 0$ there exists a $C(N)$ such, that for all sufficiently small positive t

$$(4.13) \quad |K_t^F(x, y)| \leq C(N) \frac{t^{-d}}{\max(1, t^{-1}|x - y|)^N}, \quad t > 0.$$

For such kind estimates see [9], [13], [14], [17], [29].

Let's return to our frame $\{\Theta_{j,i}\}$. One has,

$$|\Theta_{j,i}(x)| = |F(2^{-2j}L)\Phi_{j,i}(x)| = \left| \int_{U_{j,i}} K_{2^{-j}}^F(x, y) \Phi_{j,i}(y) dy \right| \leq \sup_{y \in U_{j,i}} |K_{2^{-j}}^F(x, y)| \leq C(N) \sup_{y \in U_{j,i}} \frac{2^{dj}}{\max(1, 2^j|x - y|)^N}.$$

Thus, the following statement about localization of every $\Theta_{j,i}$ holds.

Lemma 4.2. *For any $N > 0$ there exists a $C(N)$ such, that*

$$(4.14) \quad |\Theta_{j,i}(x)| \leq C(N) \sup_{y \in U_{j,i}} \frac{2^{dj}}{\max(1, 2^j|x - y|)^N}, \quad j \in [J, \infty),$$

uniformly in i and j .

Inequality (4.7) and Lemma 4.2 give the Frame Theorem 1.1.

5. BESOV SPACES

Let L be a self-adjoint positive definite operator in a Hilbert space $L_2(\Omega)$ which was introduced in the first section. We consider its positive root $L^{1/2}$ and let $\mathcal{D}_r, r \in \mathbb{R}$, be the domain of the operator $L^{r/2}, r \in \mathbb{R}$.

The inhomogeneous Besov space $B_q^\alpha(\Omega)$ is introduced as an interpolation space between the Hilbert space $L_2(\Omega)$ and Sobolev space \mathcal{D}_r where r can be any natural number such that $0 < \alpha < r, 1 \leq q < \infty$, or $0 \leq \alpha \leq r, q = \infty$. Namely, we have

$$B_q^\alpha(\Omega) = (L_2(\Omega), \mathcal{D}_r)_{\theta, q}^K, \quad 0 < \theta = \alpha/r < 1, \quad 1 \leq q \leq \infty.$$

where K is the Peetre's interpolation functor. It is known that this Besov norm can be described in terms of a modulus of continuity constructed in terms of the wave semigroup e^{itD} . Namely, let $\alpha < r \in \mathbb{N}$. The norm of the Besov space $B_q^\alpha(\Omega)$ in $L_2(\Omega)$ is equivalent to

$$(5.1) \quad \|f\|_{L_2(\Omega)} + \left(\int_0^1 (s^{-\alpha} W_r(s, f))^q ds/s \right)^{1/q}$$

for $1 \leq q < \infty$ and equivalent to

$$\|f\|_{L_2(\Omega)} + \sup_{0 < s < 1} (s^{-\alpha} W_r(s, f))$$

for $q = \infty$, where modulus of continuity is introduced as

$$W_r(s, f) = \sup_{0 < \tau \leq s} \left\| \left(I - e^{i\tau\sqrt{L}} \right)^r f \right\|_{L_2(\Omega)}.$$

Let F be the function constructed in (4.1) and $F_j(s) = F(2^{-2j}s)$. Using (4.8) and (4.9) one can introduce projectors $\mathcal{P}_j = F_j(L)$:

$$(5.2) \quad \mathcal{P}_j : L_2(\Omega) \rightarrow E_{[2^{2j-2}, 2^{2j+4}]}(L), \quad \|\mathcal{P}_j\| \leq 1.$$

According to (4.2) the following version of Calderón decomposition holds:

$$(5.3) \quad \sum_{j \in \mathbb{N}} |F_j|^2(L) f = f, \quad f \in L_2(L),$$

where the series converges in $L_2(\Omega)$.

Theorem 5.1. *The norm of the Besov space $B_q^\alpha(\Omega)$ for $\alpha > 0, 1 \leq q \leq \infty$ is equivalent to*

$$(5.4) \quad \left(\sum_{j=0}^{\infty} (2^{j\alpha} \|f_j(f)\|_{L_2(\Omega)})^q \right)^{1/q}, \quad f_j(f) = \mathcal{P}_j f,$$

with the standard modifications for $q = \infty$.

Let $\{\Theta_{j,i}\}$ be the frame in $L_2(\Omega)$ defined in (4.6). Theorem 5.1 along with (4.7) immediately imply the following description of Besov norm in terms of frame coefficients.

Theorem 5.2. *The norm of the Besov space $B_q^\alpha(\Omega)$ for $\alpha > 0, 1 \leq q \leq \infty$ is equivalent to*

$$(5.5) \quad \left(\sum_{j=0}^{\infty} 2^{j\alpha q} \left(\sum_{i=1}^{I_j} |\langle f, \Theta_{j,i} \rangle|^2 \right)^{q/2} \right)^{1/q},$$

with the standard modifications for $q = \infty$.

The Theorem 5.1 will be obtained as a consequence of the two lemmas below. First we note the following inequalities hold for $j \in \mathbb{N}$

$$(5.6) \quad \left\| \left(I - e^{i\tau\sqrt{L}} \right)^r \mathcal{P}_j \right\| \leq C_j \tau^r 2^{(j+1)r/2},$$

where $\tau > 0$ and r is a natural number. These inequalities follow from general properties of semigroups of operators and from the properties (5.2).

Let $r \in N$, $m, k \in \mathbb{R}$ such that $k + m \geq 0$ and $k < 0$. Define $w_j = 2^{kj}$ and $c_j = 2^{mj}$ for $j \in \mathbb{Z}$. Using these notations and previous two inequalities one can easily verify, that there exists a C such that for any $f \in L_2(\Omega)$ and any $1 \leq p < \infty$ the following holds

$$(5.7) \quad (W_r(s, f))^p \leq C \left(s^{pr} \sum_{j=0}^{\infty} \left(2^{jpr/2} w_j c_j^p \|f_j\|_{L_2(\Omega)}^p \right) \right), \quad s \in (0, 1].$$

Lemma 5.3. *If for an $f \in L_2(\Omega)$ the following condition is satisfied*

$$(5.8) \quad \sum_{j=0}^{\infty} 2^{j\alpha q} \|f_j\|_{L_2(\Omega)}^q < \infty, \quad f_j(f) = \mathcal{P}_j f, \quad 1 \leq q \leq \infty,$$

then f belongs to $B_q^\alpha(\Omega)$. Moreover, there exists a C for which

$$(5.9) \quad \left(\int_0^1 (s^{-\alpha} W_r(f, s))^q ds/s \right)^{1/q} \leq C \left(\sum_{j=0}^{\infty} (2^{j\alpha} \|f_j\|_{L_2(\Omega)})^q \right)^{1/q}$$

for all f that satisfy (5.8).

Proof. Pick a $1 \leq q < \infty$ and let $r \leq 2\alpha$ and k and m satisfy the inequality $k + mq \leq q(\alpha - r/2)$ (such choice is not unique.) By (5.7) we have the following inequality

$$(5.10) \quad \left(\sup_{\tau \leq s} \left\| \left(I - e^{i\tau\sqrt{L}} \right)^r f \right\|_{L_2(\Omega)} \right)^q \leq C \left\{ s^{qr} \sum_{j=0}^{\infty} 2^{jqr/2} w_j c_j^q \|f_j\|_{L_2(\Omega)}^q \right\}.$$

By integrating the both sides of (5.10) on $[0, 1]$ with respect to the measure $s^{-\alpha q} \frac{ds}{s}$ we get

$$(5.11) \quad \left(\int_0^1 (s^{-\alpha} W_r(f, s))^q ds/s \right)^{1/q} \leq C \left(\sum_{j=0}^{\infty} 2^{jqr/2} w_j c_j^q \|f_j\|_{L_2(\Omega)}^q \right)^{1/q}.$$

Next, by using triangle inequality for l_q - norm and the inequalities $2^{jqr/2} w_j c_j^q \leq 2^{j\alpha q}$ we obtain

$$(5.12) \quad \left(\sum_{j=0}^{\infty} 2^{jqr/2} w_j c_j^q \|f_j\|_{L_2(\Omega)}^q \right)^{1/q} \leq \left(\sum_{j=0}^{\infty} 2^{j\alpha q} \|f_j\|_{L_2(\Omega)}^q \right)^{1/q}.$$

This completes the proof of Lemma for $1 \leq q < \infty$. The case $q = \infty$ can be handled in a similar way. \square

Lemma 5.4. *There exists a C such that for any $f \in L_2(\Omega)$ and any $1 \leq q \leq \infty$ the following holds*

$$(5.13) \quad \left(\sum_{j=0}^{\infty} (2^{j\alpha} \|f_j\|_{L_2(\Omega)})^q \right)^{1/q} \leq C \left(\int_0^1 (s^{-\alpha} W_r(f, s))^q ds/s \right)^{1/q}.$$

Proof. In this proof a constant $\alpha > 0$ and a natural number r will be fixed. It allows us to introduce the following notation

$$a_0 = \int_0^1 s^{-\alpha} \left| 1 - e^{is/2} \right|^{4r} ds$$

without explicitly mentioning α and r . Let $1/4 \leq t \leq 1$. By substituting $s \mapsto 2s\sqrt{t}$ in the above integral we get

$$a_0 = \int_0^1 \frac{1}{2\sqrt{t}} \left(2s\sqrt{t} \right)^{-\alpha} \left| 1 - e^{is\sqrt{t}} \right|^{4r} \sqrt{t} ds.$$

Since $1/\sqrt{t} \leq 2$ and $\sqrt{t} \leq 1$, the following estimations hold for any $Q > 0$ up to some constants independent of t :

$$\begin{aligned} a_0 &= \int_0^1 \frac{1}{2\sqrt{t}} \left(2s\sqrt{t} \right)^{-\alpha} \left| 1 - e^{is\sqrt{t}} \right|^{4r} \sqrt{t} ds \leq C \int_0^1 \left(s\sqrt{t} \right)^{-\alpha} \left| 1 - e^{is\sqrt{t}} \right|^{4r} ds \leq \\ (5.14) \quad & C \int_0^1 s^{-\alpha} \left(\sqrt{t} \right)^{-(\alpha+Q)} \left| 1 - e^{is\sqrt{t}} \right|^{4r} ds. \end{aligned}$$

Now define

$$H(t) := \int_0^1 s^{-\alpha} \left(\sqrt{t} \right)^{-(\alpha+Q)} \left| 1 - e^{is\sqrt{t}} \right|^{4r} ds$$

on $1/4 \leq t \leq 1$ and zero elsewhere. The map H is bounded and $H(t) \geq a_0$. By the spectral theory for L the operator $H(L)$ is bounded on $L_2(\Omega)$ and we obtain the following in the weak sense.

$$I \leq a_0^{-1} \int_0^1 s^{-\alpha} \left(\sqrt{L} \right)^{-(\alpha+Q)} \left| 1 - e^{is\sqrt{L}} \right|^{4r} ds$$

where I is the identity operator on the Hilbert space $L_2(\Omega)$. Therefore for any $g \in L_2(\Omega)$

$$\|g\|_{L_2(\Omega)}^2 \leq \int_0^1 s^{-\alpha} \left\| \left(\sqrt{L} \right)^{-(\alpha+Q)/2} \left(1 - e^{is\sqrt{L}} \right)^{2r} g \right\|_{L_2(\Omega)}^2 ds.$$

Take $g := f_j$, $j \geq 0$. An application of inequalities (5.6), (5.7), (5.2), and (3.18) gives

$$\begin{aligned} \|f_j\|_{L_2(\Omega)}^2 &\leq \int_0^1 s^{-\alpha} \left\| \left(\sqrt{L} \right)^{-(\alpha+Q)/2} \left(1 - e^{is\sqrt{L}} \right)^{2r} f_j \right\|_{L_2(\Omega)}^2 ds \\ &\leq \int_0^1 s^{-\alpha} \left\| \left(\sqrt{L} \right)^{-(\alpha+Q)/2} \left(1 - e^{is\sqrt{L}} \right)^r \mathcal{P}_j \right\|_{L_2(\Omega)} \left\| \left(1 - e^{is\sqrt{L}} \right)^r f \right\|_{L_2(\Omega)}^2 ds \\ &\leq C \int_0^1 s^{-\alpha} W_r(s, f)^2 \left(2^{-j(\alpha+Q)/2} 2^{jr} \right) s^{2r} ds \\ &= 2^{-j(\alpha+Q-2r)/2} \int_0^1 \left(s^{-\alpha} W_r(s, f) \right)^2 s^{2r+\alpha} ds. \end{aligned}$$

The reverse Hölder inequality for $p = 2$ for functions $\mathcal{F}(s) := s^{-\alpha}W_r(s, f)$, $\mathcal{G}(s) := 1$, $0 < s \leq 1$, gives

$$\|\mathcal{F}\|_{L^2}\|\mathcal{G}\|_{L^{-1}} \leq \|\mathcal{F}\mathcal{G}\|_{L^1}, \quad \|\mathcal{F}\|_{L^2} \leq \|\mathcal{G}\|_{L^1}\|\mathcal{F}\mathcal{G}\|_{L^1}.$$

In other words, one has

$$\begin{aligned} \int_0^1 (s^{-\alpha}W_r(s, f))^2 s^{r+\alpha} ds &\leq \left(\int_0^1 s^{2r+\alpha} ds \right)^2 \left(\int_0^1 s^{-\alpha}W_r(s, f) s^{2r+\alpha} ds \right)^2 = \\ &c \left(\int_0^1 s^{-\alpha}W_r(s, f) s^{2r+\alpha} ds \right)^2, \end{aligned}$$

for $c = c(\alpha, r) = (2r + \alpha + 1)^{-2}$. After all we get

$$\|f_j\|_{L_2(\Omega)} \leq 2^{-j(\alpha+Q-2r)/4} \int_0^1 \mathcal{F}(s) s^{2r+\alpha} ds.$$

Therefore for $1 < q < \infty$ and $q' = \frac{q}{q-1}$

$$\begin{aligned} \|f_j\|_{L_2(\Omega)} &\leq c 2^{-j(\alpha+Q-2r)/4} \int_0^1 \mathcal{F}(s) s^{2r+\alpha} ds \leq \\ &c 2^{-j(\alpha+Q-2r)/4} \left\{ \int_0^1 |\mathcal{F}(s)|^q s^{2r+\alpha} ds \right\}^{1/q} \left\{ \int_0^1 s^{2r+\alpha} ds \right\}^{1/q'} \leq \\ (5.15) \quad &c' 2^{-j(\alpha+Q-2r)/4} \left\{ \int_0^1 (s^{-\alpha}W_r(s, f))^q s^{2r+\alpha} ds \right\}^{1/q}, \end{aligned}$$

with $c' = c'(r, \alpha) = (r + \alpha + 1)^{-2+1/q'}$. Thus,

$$(5.16) \quad \|f_j\|_{L_2(\Omega)}^q \leq c 2^{-jq(\alpha+Q-2r)/4} \left(\int_0^1 (s^{-\alpha}W_r(s, f))^q ds/s \right),$$

and then, for $Q > 3\alpha + 2r$ we have

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{j\alpha q} \|f_j\|_{L_2(\Omega)}^q &\leq \left(\sum_{j \geq 0} 2^{-jq(Q-3\alpha-2r)/4} \right) \left(\int_0^1 (s^{-\alpha}W_r(s, f))^q ds/s \right) = \\ &c \left(\int_0^1 (s^{-\alpha}W_r(s, f))^q ds/s \right). \end{aligned}$$

Finally, we obtain, that for a certain $C > 0$ the following inequality holds

$$\left(\sum_{j=0}^{\infty} 2^{j\alpha q} \|f_j\|_{L_2(\Omega)}^q \right)^{1/q} \leq C \left(\int_0^1 (s^{-\alpha}W_r(s, f))^q s^r ds/s \right)^{1/q}.$$

This completes the proof of the lemma for $1 < q < \infty$. A proof for $q = 1$ and $q = \infty$ can be obtained from the preceding calculations. \square

REFERENCES

- [1] R.A. Adams, *Sobolev spaces*, Academic Press, 1978.
- [2] P.Baldi, G.Kerkycharian, D.Marinucci, D.Picard, *Asymptotics for Spherical Needlets*, *Annals of Statistics*, Vol. 37, (2009), No. 3, 1150-1171
- [3] M. Calixto, J. Guerrero, J. C. Sánchez-Monreal, *Sampling theorem and discrete Fourier transform on the hyperboloid*, *J. Fourier Anal. Appl.* 17 (2011), no. 2, 240-264.
- [4] J. Christensen, G. Olafsson, *Examples of coorbit spaces for dual pairs*, *Acta Appl. Math.* 107 (2009), no. 1-3, 25-48
- [5] J. Christensen, G. Olafsson, *Coorbit spaces for dual pairs*, *Appl. Comput. Harmon. Anal.* 31 (2011), no. 2, 303-324.
- [6] T. Coulhon, G. Kerkycharian, P. Petrushev, *Heat kernel generated frames in the setting of Dirichlet spaces*, arXiv:1206.0463.
- [7] H. Feichtinger, I. Pesenson, *Recovery of band-limited functions on manifolds by an iterative algorithm*, in: *Wavelets, Frames and Operator Theory*, in: *Contemp. Math.*, vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 137-152.
- [8] H.G. Feichtinger, I. Pesenson, *A reconstruction method for band-limited signals on the hyperbolic plane*, *Sampl. Theory Signal Image Process.* 4 (2) (2005) 107-119.
- [9] F. Filbir, H. Mhaskar, *A quadrature formula for diffusion polynomials corresponding to a generalized heat kernel*, *J. Fourier Anal. Appl.* 16 (2010), no. 5, 629-657.
- [10] F. Filbir, H.N. Mhaskar, *Marcinkiewicz-Zygmund measures on manifolds*, *J. Complexity* 27 (2011), no. 6, 568-596.
- [11] H. Führ, A. Mayeli, *Homogeneous Besov spaces on stratified Lie groups and their wavelet characterization*, will appear in *J. Function spaces and Applications*, 2012.
- [12] H. Führ, *Abstract harmonic analysis of continuous wavelet transforms*, *Lecture Notes in Mathematics*, 1863. Springer-Verlag, Berlin, 2005. x+193 pp. ISBN: 3-540-24259-7
- [13] D. Geller and A. Mayeli, *Nearly Tight Frames and Space-Frequency Analysis on Compact Manifolds* (2009), *Math. Z.* **263** (2009), 235-264.
- [14] D. Geller and A. Mayeli, *Besov spaces and frames on compact manifolds*, *Indiana Univ. Math. J.* 58 (2009), no. 5, 2003-2042.
- [15] D. Geller and D. Marinucci, *Mixed needlets*, *J. Math. Anal. Appl.* 375 (2011), no. 2, 610-630.
- [16] D. Geller, and D. Marinucci, (2010) *Spin Wavelets on the Sphere*, *Journal of Fourier Analysis and its Applications*, Vol. 6, pp.840-884, arxiv: 0811.2935.
- [17] D. Geller and I. Pesenson, *Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds*, *J. Geom. Anal.* 21 (2011), no. 2, 334-37.
- [18] D. Geller, A. Mayeli, *Wavelets on manifolds and statistical applications to cosmology*, *Wavelets and multiscale analysis*, 259-277, *Appl. Numer. Harmon. Anal.*, Birkhuser/Springer, New York, 2011.
- [19] L. Hörmander, *The analysis of linear partial differential operators. III. Pseudo-differential operators*, Springer, Berlin, 2007. viii+525 pp. ISBN: 978-3-540-49937-4.
- [20] B. Leistedt, J.D. McEwen, *Exact Wavelets on the Ball*, arXiv:1205.0792.
- [21] F. Lanusse, A. Rassat, and J.-L. Starck, *Spherical 3D isotropic wavelets*, *Astronomy and Astrophysics*, 540, A92 (2012).
- [22] J. L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problem and Applications*, Springer-Verlag, 1975.
- [23] S. Mallat, *Group Invariant Scattering*, to appear in *Communications in Pure and Applied Mathematics*, 2012, <http://arxiv.org/abs/1101.2286>.
- [24] D. Marinucci, G. Peccati, *Random fields on the sphere. Representation, limit theorems and cosmological applications*, *London Mathematical Society Lecture Note Series*, 389. Cambridge University Press, Cambridge, 2011. xii+341 pp. ISBN: 978-0-521-17561-6.
- [25] D. Marinucci, D. Pietrobon, A. Balbi, P. Baldi, P. Cabella, G. Kerkycharian, P. Natoli, D. Picard, N. Vittorio, *Spherical Needlets for CMB Data Analysis*, *Monthly Notices of the Royal Astronomical Society*, Vol. 383, (2008), pp. 539-545.
- [26] J. Marzo, *Marcinkiewicz-Zygmund inequalities and interpolation by spherical harmonics*, *J. Funct. Anal.* 250 (2007), no. 2, 559-587.
- [27] A. Mayeli, *Shannon multiresolution analysis on the Heisenberg group*, *J. Math. Anal. Appl.* 348 (2008), no. 2, 671-684.

- [28] A. Mayeli, I. Pesenson, *Space-Frequency localized wavelets for spherical Besov spaces on the Heisenberg group*, submitted.
- [29] F.J. Narcowich, P. Petrushev and J. Ward, *Localized Tight frames on spheres*, SIAM J. Math. Anal. 38, (2006), 574-594.
- [30] J. Ortega-Cerdá, B. Pridhnani, *Beurling-Landau's density on compact manifolds*, J. of Functional Analysis, Volume 263, Issue 7, (2012), 2102-2140.
- [31] I. Pesenson, *Sampling of Paley-Wiener functions on stratified groups*, J. of Fourier Analysis and Applications 4 (1998), 269-280.
- [32] I. Pesenson, *A sampling theorem on homogeneous manifolds*, Trans. Amer. Math. Soc. **352** (2000), no. 9, 4257-4269.
- [33] I. Pesenson, *An approach to spectral problems on Riemannian manifolds*, Pacific J. of Math. Vol. 215(1), (2004), 183-199.
- [34] I. Pesenson, *Poincare-type inequalities and reconstruction of Paley-Wiener functions on manifolds*, J. of Geometric Analysis, (4), 1, (2004), 101-121.
- [35] I. Pesenson, *Variational splines on Riemannian manifolds with applications to integral geometry*, Adv. in Appl. Math. 33 (2004), no. 3, 548-572.
- [36] I. Pesenson, *Deconvolution of band limited functions on symmetric spaces*, Houston J. of Math., 32, No. 1, (2006), 183-204.
- [37] I. Pesenson, *Frames in Paley-Wiener spaces on Riemannian manifolds*, in Integral Geometry and Tomography, Contemp. Math., 405, AMS, (2006), 137-153.
- [38] I. Pesenson, *A Discrete Helgason-Fourier Transform for Sobolev and Besov functions on noncompact symmetric spaces*, Contemp. Math, 464, AMS, (2008), 231-249.
- [39] I. Pesenson, *Paley-Wiener approximations and multiscale approximations in Sobolev and Besov spaces on manifolds*, J. of Geometric Analysis, 4, (1), (2009), 101-121.
- [40] I. Pesenson, M. Pesenson, *Approximation of Besov vectors by Paley-Wiener vectors in Hilbert spaces*, Approximation Theory XIII: San Antonio 2010 (Springer Proceedings in Mathematics), by Marian Neamtu and Larry Schumaker, 249-263.
- [41] I. Pesenson, *Multiresolution Analysis on compact Riemannian manifolds*, in Multiscale Analysis and Nonlinear Dynamics, M. Pesenson, Ed. New York: Wiley, 2012, p. (to be published).
- [42] P. Petrushev, Y. Xu, *Localized polynomial frames on the ball*, Constr. Approx. 27 (2008), no. 2, 121-148.
- [43] M. Taylor, *Pseudodifferential Operators*, Princeton University Press, 1981.